

APPLICATIONS OF THE TUMURA-CLUNIE THEOREM

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Abstract. Some applications of the Tumura-Clunie theorem are given. Most of these concerned fixed points of compositions of entire functions.

I. Introduction. The purpose of this note is to develop some applications of the Tumura-Clunie theory (Hayman [2, p. 69]). It is assumed that the reader is familiar with the basic quantities of Nevanlinna Theory: $T(r, f)$, $m(r, f)$, $N(r, f)$, ... and their elementary properties.

II. Notation, terminology. 1. f will always denote a nonconstant meromorphic function.

2. We shall denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$, as $r \rightarrow +\infty$, possibly outside a set of r of finite measure.

3. We shall denote by $a(z)(a_1(z), \dots)$ meromorphic functions satisfying $T(r, a(z)) = S(r, f)$.

4. By a differential polynomial $P(f)$ we understand a polynomial in $f(z)$ and its derivatives whose coefficients are of the form $a(z)$.

III. Statement and applications of the Tumura-Clunie theorem.

THEOREM 1 (TUMURA-CLUNIE [2]). Suppose $g(z) = f^n(z) + P_{n-1}(f)$, and that

$$(1) \quad N(r, f) + N(r, 1/g) = S(r, f).$$

Then $g(z) = h^n(z)$, $h(z) = f(z) + a(z)/n$ and $h^{n-1}(z)a(z)$ is obtained by substituting $h(z)$ for $f(z)$, $h'(z)$ for $f'(z)$, etc. in the terms of degree $n-1$ in $P_{n-1}(f)$.

REMARK. The conclusion still holds good if the condition (1) is replaced by:

$$N(r, f) + N(r, 1/g) = S_0(r, f),$$

where $S_0(r, f)$ denotes any quantity which satisfies $S_0(r, f) = o\{T(r, f)\}$ as $r \rightarrow +\infty$ through a set of r of infinite measure.

We have

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COROLLARY (1-1). Let $A(z)$, $B(z)$ be entire functions, $f(z) = e^{A(z)}$, $T(r, f) = O(T(r, e^{B(z)}))$. Then the identity

$$(1-1) \quad a_1(z)e^{A(z)} + a_2(z)e^{B(z)} = a_3(z)$$

cannot hold for an a_3 which is not identically equal to 0.

Proof. The result follows immediately if $a_1(z) \equiv 0$ or $a_2(z) \equiv 0$. Therefore we may assume that both $a_1(z)$ and $a_2(z)$ are not identically equal to zero.

Multiplying both sides of (1-1) by $a_1(z)$, we have

$$(1-2) \quad a_1^2(z)e^{A(z)} - a_1(z)a_3(z) = -a_1(z)a_2(z)e^{B(z)}.$$

Setting $a_1(z) \exp [A(z)/2] = H(z)$, $-a_1(z)a_2(z)e^{B(z)} = g(z)$, we have

$$(1-3) \quad H^2(z) - a_1(z)a_3(z) = g(z).$$

We note that

$$(1-4) \quad \begin{aligned} N(r, H) + N(r, 1/g) &= N(r, a_1(z)) + N(r, 1/a_1(z)a_2(z)) \\ &= S(r, e^{A(z)}) = S(r, H). \end{aligned}$$

Therefore, the Tumura-Clunie theorem is applicable to the identity (1-3), and leads to

$$(1-5) \quad H^2(z) = g(z).$$

This forces $a_1(z)a_3(z) \equiv 0$, therefore $a_3(z) \equiv 0$. In order to state Theorem 2, we need

DEFINITION 1. To each function $\lambda(r)$, positive, continuous and nondecreasing on $0 < r < +\infty$, we associate the class F_λ of meromorphic functions f satisfying $T(r, f) = O(\lambda(r))$, as $r \rightarrow \infty$ for all r except a set of r of finite measure. It is easily verified that F_λ is a field.

THEOREM 2. Let f be a meromorphic function with

$$(2-1) \quad N(r, f) = S(r, f)$$

satisfying

$$(2-2) \quad f^n(z) + P_{n-1}(f) = b(z),$$

where $P_{n-1}(f)$ is a differential polynomial in f of degree at most $n-1$, and all the coefficients in $P_{n-1}(f)$ belong to F_λ and $b(z) \in F_\lambda$, $b(z) \not\equiv 0$. Then $f \in F_\lambda$.

Proof. Assume the statement is false, i.e., $f \notin F_\lambda$. This implies that $T(r, b(z)) = S_0(r, f)$.

Combining this with the hypothesis on $N(r, f)$, we have

$$(2-3) \quad N(r, f) + N(r, 1/b(z)) = S_0(r, f).$$

Therefore, Theorem 1 is applicable by the remark made after Theorem 1. Thus we

obtain $(f(z) + a(z))^n = b(z)$. Combined with $T(r, a(z)) = S(r, f)$, $T(r, b(z)) = S_0(r, f)$ this leads to the contradiction $T(r, f) = S_0(r, f)$. Thus the theorem is proved.

A consequence of Theorem 2 is

COROLLARY (2-1) (HELLERSTEIN AND RUBEL [4]). F_λ is algebraically closed in the field of all meromorphic functions.

The following theorem generalizes a result of Rosenbloom [3].

THEOREM 3. Let f be a transcendental meromorphic function with $N(r, f) = S(r, f)$. Then $R(f)$ has infinitely many fixed points, where $R(z) = P(z)/Q(z)$, a rational function, and $P(z)$, $Q(z)$ are two relative prime polynomials with degree of $P(z) \geq 2$.

Proof. Assume $R(z) - z$ has finitely many fixed points. That is

$$(3-1) \quad \frac{P(f) - zQ(f)}{Q(f)}$$

has finitely many zeros.

Since $P(z)$, $Q(z)$ are relatively prime, the zeros of $R(z) - z$ are just the zeros of $P(f) - zQ(f)$. Thus

$$(3-2) \quad P(f) - zQ(f) = A(z)e^{B(z)},$$

where $A(z)$ is a polynomial and $B(z)$ is an entire function. Consequently,

$$(3-3) \quad (\alpha_0 z + \beta_0)f^n(z) + (\alpha_1 z + \beta_1)f^{n-1}(z) + \cdots + (\alpha_n z + \beta_n)f(z) + \alpha z + \beta = A(z)e^{B(z)},$$

where n is an integer ≥ 2 , α_i ($i=0, 1, 2, \dots, n$), β_j ($j=0, 1, 2, \dots$), α, β are constants and it is assumed that at least one of $\alpha_0, \beta_0 \neq 0$.

According to the Tumura-Clunie theorem, we obtain that

$$(3-4) \quad \left[f(z) + \frac{1}{n} \frac{\alpha_1 z + \beta_1}{\alpha_0 z + \beta_0} \right]^n = \frac{A(z)e^{B(z)}}{\alpha_0 z + \beta_0}.$$

Combining (3-3) and (3-4) we obtain

$$(3-5) \quad R_1(z)f^{n-1}(z) + R_2(z)f^{n-2}(z) + \cdots + R_n(z)f(z) \equiv R_0(z),$$

where $R_i(z)$, $i=1, 2, \dots, n$, are rational functions and

$$R_0(z) = \frac{\alpha z + \beta}{\alpha_0 z + \beta_0} - \left(\frac{\alpha_1 z + \beta_1}{\alpha_0 z + \beta_0} \right)^n \neq 0.$$

We conclude as before that

$$(3-6) \quad (f(z) + R(z))^{n-1} = T(z),$$

where $R(z)$, $T(z)$ are rational functions. This is impossible because f is transcendental and our proof is completed.

Along the same lines we have

THEOREM 4. Let $f(z)$, $g(z)$ be two transcendental entire functions and let $f(z)$ have finitely many fixed points. Let $A(z)$ be entire, and suppose that $A(z) \not\equiv g(z) - z$, and that $A(z)$ satisfies the condition

$$(4-1) \quad T(r, A(z)) = S(r, f).$$

(In the case $A(z) \equiv 0$ we also formally recognize that the condition (4-1) is fulfilled.) Then $f(g(z)) - A(z)$ has infinitely many fixed points.

Proof. Assume $f(g(z)) - A(z)$ has finitely many fixed points, that is

$$(4-2) \quad f(g(z)) - A(z) - z = P_1(z) \exp [Q_1(z)],$$

where $P_2(z)$ is a polynomial and $Q_2(z)$ is an entire function.

On the other hand, according to the hypothesis, we have

$$(4-3) \quad f(z) - z = P_2(z) \exp [Q_2(z)],$$

where $P_2(z)$ is a polynomial and $Q_2(z)$ is an entire function. Consequently,

$$(4-4) \quad f(g(z)) - g(z) = P_2(g) \exp [Q_2(g)].$$

Combining (4-2) and (4-4) we obtain

$$(4-5) \quad g(z) - z - A(z) = P_1(z) \exp [Q_1(z)] - P_2(g) \exp [Q_2(g)].$$

Since for any two entire transcendental functions f and g

$$T(r, f(g))/\{T(r, f) + T(r, g)\} \rightarrow \infty$$

(Hayman [2, pp. 50, 54]), we can apply Corollary (1-1) and conclude

$$(4-6) \quad g(z) - z - A(z) \equiv 0.$$

This contradicts the hypothesis, so our proof is completed.

REMARK. ($f(z) = e^z + z$ shows that the assumption $A(z) \not\equiv g(z) - z$ is crucial.) In the special case $A(z) \equiv 0$, we obtain another result of Rosenbloom [3].

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